

Problem 2.15

[Difficulty: 2]

2.15 A flow field is given by $\vec{V} = Ax\hat{i} + 2Ay\hat{j}$, where $A = 2 \text{ s}^{-1}$. Verify that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{2At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2, 2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 \cdot e^{A \cdot t}$ and $y_p = c_2 \cdot e^{2 \cdot A \cdot t}$; pathline that was at (2,2) at $t = 0$; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A \cdot x$ $v_p = \frac{dy}{dt} = 2 \cdot A \cdot y$

So, separating variables $\frac{dx}{x} = A \cdot dt$ $\frac{dy}{y} = 2 \cdot A \cdot dt$

Integrating $\ln(x) = A \cdot t + C_1$ $\ln(y) = 2 \cdot A \cdot t + C_2$
 $x = e^{A \cdot t + C_1} = e^{C_1} \cdot e^{A \cdot t} = c_1 \cdot e^{A \cdot t}$ $y = e^{2 \cdot A \cdot t + C_2} = e^{C_2} \cdot e^{2 \cdot A \cdot t} = c_2 \cdot e^{2 \cdot A \cdot t}$

The pathlines are $x = c_1 \cdot e^{A \cdot t}$ $y = c_2 \cdot e^{2 \cdot A \cdot t}$

Eliminating t $y = c_2 \cdot e^{2 \cdot A \cdot t} = c_2 \cdot \left(\frac{x}{c_1} \right)^2$ so $y = c \cdot x^2$ or $y = \frac{1}{2} \cdot x^2$ for given data

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{2 \cdot A \cdot y}{A \cdot x} = \frac{2 \cdot y}{x}$

So, separating variables $\frac{dy}{y} = \frac{2 \cdot dx}{x}$ Integrating $\ln(y) = 2 \cdot \ln(x) + c$

The solution is $\ln\left(\frac{y}{x^2}\right) = c$

or $y = C \cdot x^2$ or $y = \frac{1}{2} \cdot x^2$ for given data

The streamline passing through (2,2) and the pathline that started at (2,2) coincide because the flow is steady!